A LAMAN THEOREM FOR NON-EUCLIDEAN BAR-JOINT FRAMEWORKS

D. KITSON AND S. C. POWER

ABSTRACT. The minimal infinitesimal rigidity of bar-joint frameworks in the non-Euclidean spaces $(\mathbb{R}^2, \|\cdot\|_q)$ for $1 \leq q \leq \infty, q \neq 2$, are characterised in terms of $(2,2)$-tight graphs.

1. INTRODUCTION

It is a longstanding open problem in the theory of bar-joint frameworks to obtain a form of combinatorial characterisation for the infinitesimal rigidity of generic frameworks in the Euclidean space $\mathbb{R}^3$. On the other hand in two dimensions the foundational characterisation of Laman [7] provides a necessary and sufficient condition, namely that the underlying simple graph (V, E) should contain a graph (V, E') which is $(2,3)$-tight, meaning that the Maxwell count $|E'| = 2|V| - 3$ should hold, together with the inequalities $|E(H)| \leq 2|V(H)| - 3$ for all subgraphs H with $|E(H)| \geq 1$. We show that for the non-Euclidean norms $\|\cdot\|_q, 1 \leq q \leq \infty, q \neq 2$, there are exact analogues of Laman's characterisation in terms of $(2,2)$-tight graphs and in this spirit we pose several problems on the characterisation of generic bar-joint frameworks in normed spaces.

For the theory of bar-joint frameworks in finite dimensional Euclidean space see, for example, Asimow and Roth [1], [2], Gluck [4] and Graver, Servatius and Servatius [5]. It seems that the setting of general finite-dimensional normed spaces is new and requires novel combinatorial graph theory. One can verify for example that a triangular framework (K_3, p) in $(\mathbb{R}^2, \|\cdot\|_q)$ has a flex (of rotational type) which does not derive from an ambient rigid motion and that a regular K_4 framework is minimally infinitesimally rigid. These observations echo similar facts for bar-joint frameworks in three dimensions whose vertices are constrained to a circular cylinder, and indeed the circular cylinder and $(\mathbb{R}^2, \|\cdot\|_q)$ both possess a two dimensional space of trivial infinitesimal motions. Accordingly it is K_4 (or K_1) rather than K_2 which will play the role of a base graph in the inductive scheme appropriate for this normed space.

A standard proof of Laman's theorem makes use of the two Henneberg moves $G \to G'$ which increase by one the number of vertices and which generate all $(2,3)$-tight graphs from the base graph K_2. For $q \neq 1, 2, \infty$ we take a similar approach using both an associated rigidity matrix and an inductive scheme for $(2,2)$-tight graphs in terms of the two Henneberg moves and two additional moves, namely the vertex-to-4-cycle move and the vertex-to-K_4 move. For $q \in \{1, \infty\}$ some novel features emerge in that infinitesimal rigidity is characterised by induced graph colourings and the set of regular realisations is no longer dense. We develop this idea in the broader context of polytopic norms.

2. Infinitesimal flexes

We define a bar-joint framework in the normed linear space $(X, \|\cdot\|)$ to be a pair (G, p) with G a simple graph (V, E) and $p = (p_1, p_2, \ldots, p_n)$ an n-tuple of points in X representing the placement of the vertices of V with respect to some given labelling $v_1, v_2, \ldots, v_n$ of V. In fact we may assume that the graph is connected and so we do so throughout.

Definition 2.1. An *infinitesimal flex* of (G, p) is a vector $u = (u_1, \ldots, u_n) \in X^n$ such that the corresponding framework edge lengths of $(G, p + tu)$ have second order deviation from the original lengths,

$$\|(p_i + tu_i) - (p_j + tu_j)\| - \|p_i - p_j\| = o(t), \qquad \text{as } t \to 0$$

for each edge $v_i v_j \in E$.

The infinitesimal flexes of a framework (G, p) form a proper closed linear subspace of X^n.

Definition 2.2. A *rigid motion* of the normed space $(X, \|\cdot\|)$ is a collection $\{\gamma_x\}_{x \in X}$ of continuous paths $\gamma_x : [-1, 1] \to X$ such that

    (1) γ_x is differentiable at 0 and $\gamma_x(0) = x$ for each $x \in X$,
    (2) $\|\gamma_x(t) - \gamma_y(t)\| = \|x - y\|$ for all $t \in [-1, 1]$ and for all $x, y \in X$, and,
    (3) the isometries $\Gamma_t : X \to X$, $x \mapsto \gamma_x(t)$ are surjective maps for all $t \in [-1, 1]$.

Each rigid motion gives rise to a vector field on X via the map $x \mapsto \gamma_x'(0)$ and this vector field induces an infinitesimal flex on any given framework (G, p) as shown in the following lemma. An infinitesimal flex is said to be *trivial* if it is obtained from a rigid motion in this way.

Lemma 2.3. *Let (G, p) be a bar-joint framework in a normed linear space $(X, \|\cdot\|)$.*
    (1) *If $\{\gamma_x\}_{x \in X}$ is a rigid motion then $u = (\gamma_{p_1}'(0), \ldots, \gamma_{p_n}'(0))$ is a (trivial) infinitesimal flex of (G, p).*
    (2) *If $a \in X$ then $u = (a, \ldots, a) \in X^n$ is a trivial infinitesimal flex of (G, p).*

Proof. (i) For each edge $v_i v_j \in E$ we compute

$$
\begin{aligned}
\epsilon(t) &= \left| \frac{\|(p_i + tu_i) - (p_j + tu_j)\| - \|p_i - p_j\|}{t} \right| \\
&= \left| \frac{\|(\gamma_{p_i}(0) + t\gamma_{p_i}'(0)) - (\gamma_{p_j}(0) + t\gamma_{p_j}'(0))\| - \|\gamma_{p_i}(t) - \gamma_{p_j}(t)\|}{t} \right| \\
&\leq \left\| \frac{(\gamma_{p_i}(t) - \gamma_{p_j}(t)) - (\gamma_{p_i}(0) - \gamma_{p_j}(0))}{t} - (\gamma_{p_i}'(0) - \gamma_{p_j}'(0)) \right\|
\end{aligned}
$$

From the latter quantity we see that $\epsilon(t) \to 0$ as $t \to 0$ and so u is an infinitesimal flex.

(ii) Clearly $u = (a, \ldots, a)$ is an infinitesimal flex and is induced by the rigid motion $\{\gamma_x\}_{x \in X}$ with $\gamma_x(t) = x + at$ for each $x \in X$. $\qquad\square$

In the case of the ℓ^q norms and polytopic norms on $\mathbb{R}^d$ considered in Sections 3 and 4 the trivial infinitesimal flexes of a bar-joint framework are described by the following lemma.

Lemma 2.4. *Let (G, p) be a bar-joint framework in $(X, \| \cdot \|)$ where X is a normed linear space over $\mathbb{R}$ which admits only finitely many surjective linear isometries. Then u is a trivial infinitesimal flex of (G, p) if and only if $u = (a, \ldots, a) \in X^n$ for some $a \in X$.*

Proof. The sufficiency of the condition is proved in Lemma 2.3(*ii*). To prove necessity suppose that u is a trivial infinitesimal flex of (G, p) and let $\{\gamma_x\}_{x \in X}$ be a rigid motion with $u = (\gamma'_{p_1}(0), \ldots, \gamma'_{p_n}(0))$. By the Mazur-Ulam theorem, $\Gamma_t : X \to X$, $x \mapsto \gamma_x(t)$ is an affine transformation and so there exists a linear map A_t and a vector $c(t)$ with $\Gamma_t(x) = A_t(x) + c(t)$ for all $x \in X$. The linear map A_t is a surjective isometry and we claim that for small values of t, A_t is the identity. Since γ_x is continuous it follows that $c(t) \to c(0)$ and $A_t(x) \to x$ as $t \to 0$ for all $x \in X$. Let $I, T_1, \ldots, T_m$ be the finitely many surjective linear isometries on X and choose vectors $x_1, \ldots, x_m \in X$ with $T_j(x_j) \neq x_j$. If $\epsilon = \min_{j=1,\ldots,m} \|T_j(x_j) - x_j\|$ then there exists $\delta > 0$ such that $\max_{j=1,\ldots,m} \|A_t(x_j) - x_j\| < \epsilon$ for all $|t| < \delta$. Hence $A_t = I$ for all $|t| < \delta$. We conclude that $\Gamma_t(x) = x + c(t)$ for all $|t| < \delta$ and so $\gamma'_x(0) = c'(0)$ for all $x \in X$. In particular, $u = (c'(0), \ldots, c'(0)) \in X^n$. $\square$

Definition 2.5. A bar-joint framework (G, p) is *infinitesimally flexible* in $(X, \| \cdot \|)$ if it has a non-trivial infinitesimal flex. Otherwise it is said to be *infinitesimally rigid*.

A framework (G, p) is *minimally infinitesimally rigid* (or *isostatic*) if it is infinitesimally rigid and removing a single edge from G results in an infinitesimally flexible framework.

Lemma 2.6. *Let $A : X \to Y$ be an isometric affine isomorphism between normed linear spaces $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$. Then a bar-joint framework (G, p) in X is (minimally) infinitesimally rigid if and only if $(G, A(p))$ is (minimally) infinitesimally rigid in Y.*

Proof. Here we are using the obvious notation $A(p) = (A(p_1), \ldots, A(p_n))$. If a vector $u \in X^n$ is an infinitesimal flex for (G, p) then $A(u) \in Y^n$ is clearly an infinitesimal flex for $(G, A(p))$ and vice versa. There is a one-to-one correspondence between the rigid motions $\{\gamma_x\}_{x \in X}$ of X and the rigid motions $\{\tilde{\gamma}_y\}_{y \in Y}$ of Y given by $\tilde{\gamma}_{A(x)}(t) = A(\gamma_x(t))$. If $u = (\gamma'_{p_1}(0), \ldots, \gamma'_{p_n}(0))$ is a trivial infinitesimal flex of (G, p) then $A(u) = (\tilde{\gamma}'_{A(p_1)}(0), \ldots, \tilde{\gamma}'_{A(p_n)}(0))$ and so $A(u)$ is a trivial infinitesimal flex of $(G, A(p))$. We conclude that if (G, p) is infinitesimally rigid in X then $(G, A(p))$ is infinitesimally rigid in Y. $\square$

3. ℓ^q NORMS

The aim of this section is to introduce the rigidity matrix for a framework (G, p) in $(\mathbb{R}^d, \| \cdot \|_q)$ and to characterise the minimally infinitesimally rigid frameworks in $(\mathbb{R}^2, \| \cdot \|_q)$ when $1 < q < \infty$ and $q \neq 2$. The 1-norm and ∞-norm are dealt with in the next section.

For a point $a = (a_1, \ldots, a_d) \in \mathbb{R}^d$ and $k \in (0, \infty)$ we write

$$a^{(k)} = (sgn(a_1)|a_1|^k, \ldots, sgn(a_d)|a_d|^k)$$

where sgn is the sign function. We remark that for $1 < q < \infty$ the mapping $\mathbb{R}^d \to \mathbb{R}^d$, $a \mapsto a^{(q-1)}$ is injective. It follows that p_1, p_2, p_3 are non-collinear points in $\mathbb{R}^d$ if and only if $(p_1 - p_2)^{(q-1)}$ and $(p_1 - p_3)^{(q-1)}$ are linearly independent. We will make use of this fact later.

Definition 3.1. The *rigidity matrix* $R_q(G, p)$ is a $|E| \times nd$ matrix with rows indexed by the edges of G and nd many columns indexed by the coordinates of the vertex placements $p_1, \ldots, p_n$. The row entries which correspond to an edge $v_i v_j \in E$ are

$$\begin{bmatrix} 0 & \cdots & 0 & (p_i - p_j)^{(q-1)} & 0 & \cdots & 0 & -(p_i - p_j)^{(q-1)} & 0 & \cdots & 0 \end{bmatrix}$$

where non-zero entries may only appear in the p_i columns and the p_j columns.

Let $e_1 = (1, 0, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, etc. denote the usual basis in $\mathbb{R}^d$. A bar-joint framework (G, p) is *well-positioned* in $(\mathbb{R}^d, \| \cdot \|_q)$ if for each edge $v_i v_j \in E$ and each basis vector e_k we have $(p_i - p_j) \cdot e_k \neq 0$.

Proposition 3.2. *Let (G, p) be a well-positioned bar-joint framework in $(\mathbb{R}^d, \| \cdot \|_q)$, $1 < q < \infty$, $q \neq 2$. Then*

(1) *$u \in \mathbb{R}^{nd}$ is an infinitesimal flex for (G, p) if and only if $R_q(G, p)u = 0$.*
(2) *(G, p) is infinitesimally rigid if and only if $\operatorname{rank} R_q(G, p) = dn - d$.*

Proof. (i) Let $u = (u_1, \ldots, u_n) \in \mathbb{R}^{nd}$ and for each edge $v_i v_j \in E$ consider the associated edge length function

$$\zeta_{ij} : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \|(p_i + tu_i) - (p_j + tu_j)\|_q$$

Note that u is an infinitesimal flex for (G, p) if and only if $\zeta_{ij}'(0) = 0$ for each edge $v_i v_j \in E$. Since (G, p) is well-positioned, $\zeta_{ij}'(0)$ exists for each edge and we can compute

$$\zeta_{ij}'(0) = \frac{(p_i - p_j)^{(q-1)} \cdot (u_i - u_j)}{\|p_i - p_j\|_q^{q-1}}$$

from which the result follows.

(ii) Recall that there are only finitely many surjective linear isometries on $(\mathbb{R}^d, \| \cdot \|_q)$ for $q \in (1, \infty)$, $q \neq 2$. These are given by signed permutation matrices (see [8, Proposition 2.f.14]) and have the form $T(x) = (\theta_1 x_{\pi(1)}, \ldots, \theta_d x_{\pi(d)})$ for some permutation π of $\{1, 2, \ldots, d\}$ and some $\theta_j \in \{-1, 1\}$. Applying Lemma 2.3 we see that the trivial infinitesimal flexes form a d dimensional subspace of $\mathbb{R}^{nd}$. The result now follows from (i). $\square$

As in Euclidean space the set of all realisations of a bar-joint framework in $(\mathbb{R}^d, \| \cdot \|_q)$ can be partitioned into regular and non-regular realisations.

Definition 3.3. A framework (G, p) is *regular* in $(\mathbb{R}^d, \| \cdot \|_q)$ if the rank of its rigidity matrix $R_q(G, p)$ is maximal over all framework realisations (G, p').

Note that for $q \in (1, \infty)$ the set of all regular realisations for G is an open dense subset of $\mathbb{R}^d$. If a bar-joint framework (G, p) is well-positioned and infinitesimally rigid in $(\mathbb{R}^d, \| \cdot \|_q)$ then p is necessarily a regular realisation. If a framework (G, p) is infinitesimally flexible for all realisations p in some open subset of $\mathbb{R}^d$ then (G, p) must be infinitesimally flexible for all well-positioned realisations in $\mathbb{R}^d$.

Example 3.4. It follows from Proposition 3.2 that any well-positioned realisation of the complete graphs K_2 and K_3 in $(\mathbb{R}^2, \| \cdot \|_q)$ with $1 < q < \infty$, $q \neq 2$, must be infinitesimally flexible. In contrast by computing an appropriate rigidity matrix we see that any well-positioned regular realisation of K_4 in these spaces will be infinitesimally rigid and indeed minimally infinitesimally rigid. With our definitions note that any single vertex framework (K_1, p) is infinitesimally rigid.

Remark 3.5. That a regular K_3 framework is infinitesimally flexible may seem something of a paradox and in fact there is a measure of freedom in how one may view the flexibility of small complete graphs in finite-dimensional normed spaces. One could adopt a more intrinsic notion of infinitesimal rigidity (rather than the ambient isometry one we have given above) and assert that a framework (G, p) is infinitesimally rigid if there is no infinitesimal flex $(u_1, \ldots, u_n)$ for which $\zeta'_{ij}(0) \neq 0$ for some non-edge pair i, j. By this requirement a regular K_3 framework is infinitesimally rigid $(\mathbb{R}^2, \|\cdot\|_q)$ for vacuous reasons. On the other hand the double triangle $K_3 \sqcup_e K_3$ (resulting from a Henneberg 1-move on a triangle) is infinitesimally flexible by this definition (as well as the one above) and so K_3 is exceptional in this sense also.

We now state the main result of this section (Theorem 3.6) which is a characterisation of the minimally infinitesimally rigid bar-joint frameworks in $(\mathbb{R}^2, \|\cdot\|_q)$ for $q \in (1, \infty)$ and $q \neq 2$. Recall that a simple connected graph $G = (V, E)$ is said to be $(2, 2)$-*tight* if

   (1) $|E| = 2|V| - 2$, and,
   (2) $|E(H)| \leq 2|V(H)| - 2$ for all subgraphs $H = (V(H), E(H))$.

Theorem 3.6. *Let (G, p) be a well-positioned and regular bar-joint framework in $(\mathbb{R}^2, \|\cdot\|_q)$, $1 < q < \infty$, $q \neq 2$. Then (G, p) is minimally infinitesimally rigid if and only if G is $(2, 2)$-tight.*

The rigidity matrix will play a prominent role in the proof of Theorem 3.6 and necessity can be proved directly. To prove sufficiency we will use an inductive construction based on certain graph moves.

Definition 3.7. Let $G = (V, E)$ be a simple connected graph. A graph $G' = (V', E')$ is obtained from G by a *Henneburg 1-move* by the following process:

   (1) Adjoin a new vertex v_0 to V so that $V' = V \cup \{v_0\}$.
   (2) Choose two distinct vertices $v_1, v_2 \in V$ and adjoin the edges $v_0 v_1$ and $v_0 v_2$ to E so that $E' = E \cup \{v_0 v_1, v_0 v_2\}$.

Lemma 3.8. *Let (G, p) be a well-positioned infinitesimally rigid bar-joint framework in $(\mathbb{R}^2, \|\cdot\|_q)$, $1 < q < \infty$, $q \neq 2$. If G' is obtained from G by a Henneburg 1-move then (G', p') is infinitesimally rigid for some well-positioned realisation p'.*

Proof. Suppose the Henneburg 1-move $G \to G'$ is based on the vertices $v_1, v_2 \in V$. Choose $p' = (p_0, p_1, \ldots, p_n)$ such that p' is a well-positioned realisation of G' with p_0, p_1, p_2 not collinear. If (G', p') has an infinitesimal flex $u' = (u_0, u_1, \ldots, u_n)$ then $u = (u_1, \ldots, u_n)$ is an infinitesimal flex for (G, p). Thus u is trivial and by Lemma 2.4, $u_1 = \cdots = u_n$. Now consider the rows of the rigidity matrix $R_q(G', p')$ which correspond to the edges $v_0 v_1$ and $v_0 v_2$,

$$
(3.1) \qquad \begin{bmatrix} (p_0 - p_1)^{(q-1)} & -(p_0 - p_1)^{(q-1)} & 0 & 0 & \cdots & 0 \\ (p_0 - p_2)^{(q-1)} & 0 & -(p_0 - p_2)^{(q-1)} & 0 & \cdots & 0 \end{bmatrix}
$$

Note that $u_0 - u_1 = u_0 - u_2$ is orthogonal (in the Euclidean sense) to both $(p_0 - p_1)^{(q-1)}$ and $(p_0 - p_2)^{(q-1)}$. As remarked previously the latter vectors are linearly independent in $\mathbb{R}^2$ and so $u_0 = u_1$. We conclude that u' is trivial and so (G', p') is infinitesimally rigid. $\square$

Definition 3.9. Let $G = (V, E)$ be a simple connected graph. A graph $G' = (V', E')$ is obtained from G by a *Henneburg 2-move* by the following process:

(1) Choose an edge $v_1 v_2 \in E$ and one other vertex $v_3 \in V$.
(2) Adjoin a new vertex v_0 to V so that $V' = V \cup \{v_0\}$.
(3) Remove the edge $v_1 v_2$ from E and adjoin the edges $v_0 v_1$, $v_0 v_2$ and $v_0 v_3$ so that $E' = (E \backslash \{v_1 v_2\}) \cup \{v_0 v_1, v_0 v_2, v_0 v_3\}$.

Lemma 3.10. *Let (G, p) be a well-positioned infinitesimally rigid bar-joint framework in $(\mathbb{R}^2, \|\cdot\|_q)$, $1 < q < \infty$, $q \neq 2$. If G' is obtained from G by a Henneburg 2-move then (G', p') is infinitesimally rigid for some well-positioned realisation p'.*

Proof. The set of well-positioned infinitesimally rigid realisations of G form a dense open set in $\mathbb{R}^{2n}$ and so by replacing $p = (p_1, p_2, \ldots, p_n)$ if necessary we can assume that p_1, p_2, p_3 are non-collinear in $\mathbb{R}^2$. Choose $p' = (p_0, p_1, \ldots, p_n)$ such that (G', p') is well-positioned and p_0 is a point on the line segment joining p_1 to p_2. The respective rigidity matrices can be expressed in the following block form,

$$
R_q(G', p') = \left[\begin{array}{c|ccccc}
(p_0 - p_1)^{(q-1)} & -(p_0 - p_1)^{(q-1)} & 0 & 0 & \cdots & 0 \\
(p_0 - p_2)^{(q-1)} & 0 & -(p_0 - p_2)^{(q-1)} & 0 & \cdots & 0 \\
(p_0 - p_3)^{(q-1)} & 0 & 0 & -(p_0 - p_3)^{(q-1)} & \cdots & 0 \\
\hline
0 & & & X & &
\end{array} \right]
$$

$$
R_q(G, p) = \left[\begin{array}{cccc}
(p_1 - p_2)^{(q-1)} & -(p_1 - p_2)^{(q-1)} & 0 & \cdots & 0 \\
\hline
& & X & &
\end{array} \right]
$$

The collinearity of p_0, p_1, p_2 and non-collinearity of p_1, p_2, p_3 ensure that

$$
\operatorname{rank} R_q(G', p') = \operatorname{rank} R_q(G, p) + 2 = 2(n+1) - 2
$$

and so (G', p') is infinitesimally rigid by Proposition 3.2. $\square$

Definition 3.11. Let $G = (V, E)$ be a simple connected graph. A graph $G' = (V', E')$ is obtained from G by a *vertex to 4-cycle move* by the following process:

(1) Choose a vertex $v_1 \in V$ together with two edges $v_1 v_2, v_1 v_3 \in E$.
(2) Adjoin a new vertex v_0 to V so that $V' = V \cup \{v_0\}$ and adjoin the edges $v_0 v_2$ and $v_0 v_3$ to E.
(3) Either leave any remaining edge of the form $v_1 w \in E$ unchanged or replace it with $v_0 w$.

Lemma 3.12. *Let (G, p) be a well-positioned infinitesimally rigid bar-joint framework in $(\mathbb{R}^2, \|\cdot\|_q)$, $1 < q < \infty$, $q \neq 2$. If G' is obtained from G by a vertex to 4-cycle move then (G', p') is infinitesimally rigid for some well-positioned realisation p'.*

Proof. The well-positioned and regular realisations of G and G' form dense open subsets U and V of $\mathbb{R}^{2n}$ and $\mathbb{R}^{2(n+1)}$ respectively. It follows that there exists $p' = (p_0, p_1, p_2, \ldots, p_n) \in (\mathbb{R}^2 \times U) \cap V$ such that p_1, p_2, p_3 are non-collinear. If we set $p = (p_1, p_2, \ldots, p_n)$ then (G, p) and (G', p') are both well-positioned and regular.

Since p' is a regular realisation the rank of the rigidity matrix $R_q(G', p')$ does not increase when we replace p' with $p^* = (p_1, p_1, p_2, \ldots, p_n)$. Let $u' = (u_0, u_1, \ldots, u_n)$ be in the kernel of $R_q(G', p^*)$ and consider the rows of $R_q(G', p^*)$ which correspond to the edges $v_0 v_2$, $v_0 v_3$, $v_1 v_2$ and $v_1 v_3$,

$$\begin{bmatrix} (p_1 - p_2)^{(q-1)} & 0 & -(p_1 - p_2)^{(q-1)} & 0 & 0 & \cdots & 0 \\ (p_1 - p_3)^{(q-1)} & 0 & 0 & -(p_1 - p_3)^{(q-1)} & 0 & \cdots & 0 \\ 0 & (p_1 - p_2)^{(q-1)} & -(p_1 - p_2)^{(q-1)} & 0 & 0 & \cdots & 0 \\ 0 & (p_1 - p_3)^{(q-1)} & 0 & -(p_1 - p_3)^{(q-1)} & 0 & \cdots & 0 \end{bmatrix}$$

This sytem leads us to the orthogonality relations,

$$(p_1 - p_2)^{(q-1)} \cdot (u_0 - u_1) = 0 = (p_1 - p_3)^{(q-1)} \cdot (u_0 - u_1)$$

From our earlier remark, since p_1, p_2, p_3 are non-collinear $(p_1 - p_2)^{(q-1)}$ and $(p_1 - p_3)^{(q-1)}$ are linearly independent in $\mathbb{R}^2$ and so $u_0 = u_1$. It follows that $R_q(G, p)(u_1, \ldots, u_n) = 0$ and so $u_1 = \cdots = u_n$. That (G', p') is infinitesimally rigid is now clear from Proposition 3.2 since

$$\operatorname{rank} R_q(G', p^*) = 2(n+1) - \dim \ker R_q(G', p^*) = 2(n+1) - 2$$

$\square$

Definition 3.13. Let $G = (V, E)$ be a simple connected graph. A graph $G' = (V', E')$ is obtained from G by a *vertex-to-K_4 move* by the following process:

(1) Remove a vertex v_1 from V and adjoin four new vertices w_1, w_2, w_3, w_4 so that $V' = (V \backslash \{v_1\}) \cup \{w_1, w_2, w_3, w_4\}$.
(2) Adjoin all edges of the form $w_i w_j$ to E.
(3) Replace each edge of the form $v_1 v \in E$ with $w_j v$ for some j.

Lemma 3.14. *Let (G, p) be a well-positioned infinitesimally rigid bar-joint framework in $(\mathbb{R}^2, \| \cdot \|_q)$, $1 < q < \infty$, $q \neq 2$. If G' is obtained from G by a vertex-to-K_4 move then (G', p') is infinitesimally rigid for some well-positioned realisation p'.*

Proof. Let H be the subgraph of G' which is a copy of K_4 based on the new vertices w_1, w_2, w_3, w_4. Given any realisation $p' = (p_H, p_{G' \backslash H})$ of G' the rigidity matrix can be expressed in block form

$$(3.2) \qquad R_q(G', p') = \begin{bmatrix} R_q(H, p_H) & 0 \\ X_1(p') & X_2(p') \end{bmatrix}$$

The well-positioned and regular realisations of G, H and G' form dense open sets in $\mathbb{R}^{2n}$, $\mathbb{R}^8$ and $\mathbb{R}^{2(n+3)}$ respectively. It follows that there exists a realisation $p' = (p'_1, p'_2, p'_3, p_1, p_2, \ldots, p_n) \in \mathbb{R}^{2(n+3)}$ such that the frameworks (G, p), (H, p_H) and (G', p') are all well-positioned and regular and such that the matrix $X_2(p')$ achieves its maximal rank over all realisations. Here $p = (p_1, p_2, \ldots, p_n)$ and $p_H = (p'_1, p'_2, p'_3, p_1)$.

If (G', p') has an infinitesimal flex $u' = (u'_1, u'_2, u'_3, u_1, u_2, \ldots, u_n) \in \mathbb{R}^{2(n+3)}$ then by making a translation we can assume that $u_1 = 0$. Clearly $R_q(H, p_H)(u'_1, u'_2, u'_3, u_1) = 0$. As was remarked earlier, every well-positioned regular realisation of K_4 in $(\mathbb{R}^2, \| \cdot \|_q)$ is infinitesimally rigid and so using Lemma 2.4 we have $u'_1 = u'_2 = u'_3 = u_1 = 0$. Consider now

the ill-positioned framework (G', p^*) where $p^* = (p_1, p_1, p_1, p_1, p_2, \ldots, p_n)$ and its rigidity matrix

$$(3.3) \qquad R_q(G', p^*) = \begin{bmatrix} R_q(H, p'') & 0 \\ X_1(p^*) & X_2(p^*) \end{bmatrix}$$

Note that the rigidity matrix for (G, p) can be expressed in block form

$$(3.4) \qquad R_q(G, p) = \begin{bmatrix} X_0 & X_2(p^*) \end{bmatrix}$$

Since $X_2(p')$ has maximal rank and (G, p) is infinitesimally rigid we have

$$\dim \ker X_2(p') \leq \dim \ker X_2(p^*) = 0$$

We conclude that $u_2 = u_3 = \cdots = u_n = 0$ and so (G', p') is infinitesimally rigid. $\qquad \square$

Lemma 3.15 ([12, Theorem 3.2]). *A simple graph $G = (V, E)$ is $(2, 2)$-tight if and only if there exists a finite sequence of graphs*

$$K_1 \to G^{(1)} \to G^{(2)} \to \cdots \to G$$

such that each successive graph is obtained from the previous by either a Henneburg 1-move, a Henneburg 2-move, a vertex to 4-cycle move or a vertex-to-K_4 move.

of Theorem 3.6. Let G_0 be a spanning tree for G. It is elementary that rank $R_q(G_0, p) = |E(G_0)| = n + 1$. If (G, p) is infinitesimally rigid then rank $R_q(G, p) = 2n - 2$. Moreover, if (G, p) is minimally infinitesimally rigid then the successive addition of the edges of $E(G) \backslash E(G_0)$ to G_0 strictly increases the rank of the rigidity matrix at each step. Thus exactly $n - 1$ edges are added which gives us the top count

$$|E| = |E(G_0)| + n - 1 = 2n - 2.$$

Also, if H is a vertex-induced subgraph of G then the rows of its rigidity matrix $R_q(H, p)$ are linearly independent, by the minimal infinitesimal rigidity of (G, p). If H has at least 4 vertices then it is possible to add edges to H to create a minimally infinitesimally rigid framework (H', p). Thus we have the count $|E(H)| \leq |E(H')| = 2|V(H)| - 2$ and it follows that G is $(2, 2)$-tight.

Conversely, if G is $(2, 2)$-tight then by Lemma 3.15 G can be constructed inductively by applying finitely many graph moves $K_1 \to G^{(1)} \to G^{(2)} \to \cdots \to G$. By the previous lemmas there exist well-positioned realisations $p^{(j)}$ for each $G^{(j)}$ such that $(G^{(j)}, p^{(j)})$ is infinitesimally rigid. In particular, there exists such a realisation for G and so all well-positioned regular realisations of G are infinitesimally rigid. Let $G' = (V, E')$ be a graph obtained by removing a single edge from G. Then rank $R_q(G', p) \leq |E'| = 2n - 3$ and so (G', p') is not infinitesimally rigid by Proposition 3.2. We conclude that (G, p) is minimally infinitesimally rigid. $\qquad \square$

Remark 3.16. We conjecture that the graphs G which are generically infinitesimally rigid for the non-Euclidean spaces $(\mathbb{R}^d, \| \cdot \|_q)$ with $1 < q < \infty$ are the (d, d)-tight graphs. This result would be somewhat analogous to the characterisation of generically rigid body-bar frameworks given by Tay [13] who characterises such frameworks in all dimensions in terms of (k, k)-multigraphs (where $k = d(d + 1)/2$). We also remark that higher dimensional normed spaces, including normed spaces of matrices and infinite-dimensional spaces provide further intriguing contexts for the analysis of generalised bar-joint frameworks.

There seem to be two natural approaches for this, namely (i) the development of inductive techniques as we have done here, which lends itself well to simple (non-multigraph) graph settings, and (ii) the development of a more matroidal approach via submodular functions and spanning tree decompositions in the manner of [3][6][9][14][15][16].

4. Polytopic norms

In this section we use a graph colouring technique to characterise the minimally infinitesimally rigid bar-joint frameworks in $\mathbb{R}^2$ with respect to certain norms for which the unit ball is a convex polytope. To define such a norm choose a spanning set of distinct non-zero vectors $b_1, b_2, \ldots, b_s \in \mathbb{R}^d$ and let $\mathcal{P}$ be the polytope

$$\mathcal{P} = \bigcap_{k=1}^{s} \{a \in \mathbb{R}^d : |a \cdot b_k| \le 1\}$$

The associated polytopic norm on $\mathbb{R}^d$ is $\|a\|_{\mathcal{P}} = \max_{1 \le k \le s} |a \cdot b_k|$. As special cases we obtain both the max norm and the 1-norm.

For each $a = (a_1, \ldots, a_d) \in \mathbb{R}^d$ we define $\kappa(a) = b_k$ if there is a unique index k such that $\|a\|_{\mathcal{P}} = |a \cdot b_k|$ and $\kappa(a) = 0 \in \mathbb{R}^d$ otherwise. Let (G, p) be a bar-joint framework in the normed space $(\mathbb{R}^d, \| \cdot \|_{\mathcal{P}})$.

Definition 4.1. The *rigidity matrix* $R_{\mathcal{P}}(G, p)$ is a $|E| \times nd$ matrix with rows indexed by the edges of G and nd many columns indexed by the coordinates of the vertex placements $p_1, \ldots, p_n$. The row entries which correspond to an edge $v_i v_j \in E$ are

$$\begin{bmatrix} 0 & \cdots & 0 & \kappa(p_i - p_j) & 0 & \cdots & 0 & -\kappa(p_i - p_j) & 0 & \cdots & 0 \end{bmatrix}$$

where non-zero entries may only appear in the p_i columns and the p_j columns.

As for the ℓ^q norms we say that a bar-joint framework in $(\mathbb{R}^d, \| \cdot \|_{\mathcal{P}})$ is *regular* if the rank of the rigidity matrix $R_{\mathcal{P}}(G, p)$ is maximal over all realisations. Note that the set of regular realisations of a bar-joint framework in $(\mathbb{R}^d, \| \cdot \|_{\mathcal{P}})$ form an open set but in contrast to the ℓ^q spaces it is no longer the case that the regular realisations are dense in $\mathbb{R}^d$.

Suppose (G, p) satisfies the condition that for each edge $v_i v_j \in E$ there is a unique index $k \in \{1, 2, \ldots, s\}$ such that $\|p_i - p_j\|_{\mathcal{P}} = |(p_i - p_j) \cdot b_k|$. In this case we say the edge $v_i v_j$ has framework colour k and that the framework (G, p) is *well-positioned* in $(\mathbb{R}^d, \| \cdot \|_{\mathcal{P}})$.

Proposition 4.2. *Let (G, p) be a well-positioned bar-joint framework in $(\mathbb{R}^d, \| \cdot \|_{\mathcal{P}})$. Then*
  (1) *$u \in \mathbb{R}^{nd}$ is an infinitesimal flex for (G, p) if and only if $R_{\mathcal{P}}(G, p)u = 0$.*
  (2) *(G, p) is infinitesimally rigid if and only if* rank $R_{\mathcal{P}}(G, p) = dn - d$.

Proof. The reasoning is the same as the proof of Proposition 3.2 except that in this case for each edge $v_i v_j \in E$ we compute $\zeta'_{ij}(0) = (u_i - u_j) \cdot b_k$ where k is the framework colour of the edge $v_i v_j$. $\square$

For example the above proposition shows that any well-positioned realisation of K_2 and K_3 in $(\mathbb{R}^2, \| \cdot \|_{\mathcal{P}})$ is infinitesimally flexible.

If (G, p) is well-positioned then we call the induced edge colouring $\lambda_p : E \to \{1, 2, \ldots, s\}$, $v_i v_j \to k$ where $\kappa(p_i - p_j) = b_k$ the *framework colouring* of G. We denote by G_k the largest monochrome subgraph of G for which each edge has framework colour k.

Proposition 4.3. *Let (G, p) be a well-positioned bar-joint framework in $(\mathbb{R}^d, \| \cdot \|_{\mathcal{P}})$ such that the framework colouring of G only involves the colours $1, 2, \ldots, m$ where $m \leq d$. If (G, p) is infinitesimally rigid then each of the monochrome subgraphs $G_1, G_2, \ldots, G_m$ contains a spanning tree.*

Proof. Let $k \in \{1, 2, \ldots, m\}$ and suppose that G_k does not span the vertex set of G. Then there exists a partition $V = V_1 \cup V_2$ for which there is no k-coloured connecting edge. Let W be the $(m-1)$-dimensional subspace of $\mathbb{R}^d$ spanned by $\{b_i : i = 1, \ldots, m, i \neq k\}$ and choose a non-zero vector $z \in W^{\perp}$. Then define $u = (u_1, \ldots, u_n) \in \mathbb{R}^{nd}$ such that $u_j = 0$ if $v_j \in V_1$ and $u_j = z$ if $v_j \in V_2$. We claim that u is a non-trivial infinitesimal flex. Non-triviality is clear by Lemma 2.4 since u has both zero and non-zero components. Also the flex condition is satisfied for edges with both vertices in V_1 or in V_2. For a connecting edge, $v_1 v_2$ say, suppose the framework colour is $l (\neq k)$. We have $u_1 = 0$ and $u_2 = z$ and for small enough values of t,

$$\|(p_1 + t u_1) - (p_2 + t u_2)\|_{\mathcal{P}} = |((p_1 - p_2) - tz) \cdot b_l| = \|p_1 - p_2\|_{\mathcal{P}}$$

This implies that u is a non-trivial infinitesimal flex which contradicts our hypothesis. We conclude that G_k contains a spanning tree. $\square$

Proposition 4.4. *Let (G, p) be a well-positioned bar-joint framework in $(\mathbb{R}^d, \| \cdot \|_{\mathcal{P}})$ such that the framework colouring of G involves the colours $1, 2, \ldots, m$ where $m \geq d$. If the monochrome subgraphs $G_1, G_2, \ldots, G_d$ each contain a spanning tree then (G, p) is infinitesimally rigid.*

Proof. Suppose $u = (u_1, \ldots, u_n) \in \mathbb{R}^{nd}$ is an infinitesimal flex for (G, p). Then $R_{\mathcal{P}}(G, p) u = 0$ and from the rigidity matrix we see that if $G_1, G_2, \ldots, G_d$ each contain a spanning tree then $(u_i - u_j) \cdot b_k = 0$ for each pair i, j and each k. We conclude that the only infinitesimal flexes of (G, p) are trivial. $\square$

Example 4.5. Let $\mathcal{P}$ be a polytope in $\mathbb{R}^2$ defined by linearly independent vectors $b_1, b_2 \in \mathbb{R}^2$. Then every well-positioned and regular realisation of K_4 in $(\mathbb{R}^2, \| \cdot \|_{\mathcal{P}})$ is minimally infinitesimally rigid. To see this consider the realisation $p = (p_1, p_2, p_3, p_4)$ where $p_1 = 0$, $p_2 = b_1$, $p_3 = b_1 + (1 - \epsilon) b_2$ and $p_4 = (1 + \epsilon) b_2$. Note that if ϵ is sufficiently small and non-zero then the edges $w_1 w_2$, $w_1 w_3$ and $w_3 w_4$ have framework colour 1 and the edges $w_1 w_4$, $w_2 w_3$ and $w_2 w_4$ have framework colour 2. By the above proposition (K_4, p) is infinitesimally rigid and minimally infinitesimally rigid by Proposition 4.2.

Definition 4.6. Let $G = (V, E)$ be a simple connected graph. A graph $G' = (V', E')$ is obtained from G by a *vertex splitting move* by the following process:

(1) Choose an edge $v_1 v_2 \in E$.
(2) Adjoin a new vertex v_0 to V so that $V' = V \cup \{v_0\}$ and adjoin the edges $v_0 v_1$ and $v_0 v_2$ to E.
(3) Either leave any edge of the form $v_1 u \in E$ unchanged or replace it with the edge $v_0 u$.

To characterise the minimally infinitesimally rigid bar-joint frameworks in $(\mathbb{R}^2, \| \cdot \|_{\mathcal{P}})$ we will use the following result. The Henneburg 1-move, Henneburg 2-move and vertex-to-K_4 move were defined in Section 3.

Lemma 4.7 (([10, Theorem 1.5])). *A simple graph $G = (V, E)$ is $(2, 2)$-tight if and only if there exists a finite sequence of graphs*

$$K_1 \to G^{(1)} \to G^{(2)} \to \cdots \to G$$

such that each successive graph is obtained from the previous by either a Henneburg 1-move, a Henneburg 2-move, a vertex splitting move or a vertex-to-K_4 move.

Lemma 4.8. *Let $\mathcal{P}$ be a polytope in $\mathbb{R}^2$ defined by linearly independent vectors $b_1, b_2 \in \mathbb{R}^2$. Let (G, p) be a well-positioned bar-joint framework in $(\mathbb{R}^2, \|\cdot\|_{\mathcal{P}})$ such that the monochrome subgraphs G_1 and G_2 span the vertex set of G. If G' is obtained from G by a Henneburg 1-move, a Henneburg 2-move, a vertex splitting move or a vertex-to-K_4 move then there exists a well-positioned realisation (G', p') such that the monochrome subgraphs G'_1 and G'_2 each span the vertex set of G'.*

Proof. If G' is obtained from G by a Henneberg 1-move then we can choose $p' = (p_0, p_1, \ldots, p_n)$ where p_0 lies within a sufficiently small distance of the intersection of the lines $p_1 + tb_1$ and $p_2 + tb_2$. The new edges v_0v_1 and v_0v_2 in G' have framework colours 1 and 2 respectively and all other framework colours are unchanged. Thus G' is spanned by both monochrome subgraphs.

Suppose we apply a Henneburg 2-move to G based on the edge $v_1v_2 \in E$ and the vertex $v_3 \in V$. Without loss of generality we can assume that v_1v_2 has framework colour 1. Choose $p' = (p_0, p_1, \ldots, p_n)$ where p_0 lies within a sufficiently small distance of the intersection of the line through v_1 and v_2 with the line $v_3 + tb_2$. The three new edges v_0v_1, v_0v_2 and v_0v_3 have framework colours 1, 1 and 2 respectively. All other framework colours are unchanged and so G' is spanned by both monochrome subgraphs.

Suppose that a vertex splitting move is applied to G based on the edge $v_1v_2 \in E$ and the vertex $v_1 \in V$. We assume without loss of generality that v_1v_2 has framework colour 1. Choose $p' = (p_0, p_1, p_2, \ldots, p_n)$ where $p_0 = p_1 + \epsilon b_2$. For sufficiently small non-zero values of ϵ the new edges v_0v_1, v_0v_2 in G' have framework colours 2 and 1 respectively. Each edge in G of the form v_1u is either left unchanged or is replaced with the edge v_0u. In either case the framework colouring is unchanged provided ϵ is sufficiently small. The framework colours are also unchanged for all remaining edges and so G' is spanned by the monochrome subgraphs G'_1 and G'_2.

If a vertex-to-K_4 move is applied to G based on the vertex $v_1 \in V$ then we can choose a placement for the new vertices of G' in two steps: Firstly, we arrange for the placement of K_4 to have two monochrome spanning trees. We described such a placement in Example 4.5. Secondly, we scale this placement of K_4 by a factor r so that all of its vertices are placed in a small neighbourhood of p_1. Choose $p' = (p'_1, p'_2, p'_3, p'_4, p_2, \ldots, p_n)$ where $p'_1 = p_1$, $p'_2 = p_1 + rb_1$, $p'_3 = p'_2 + r(1 - \epsilon)b_2$, $p'_4 = p_1 + r(1 + \epsilon)b_2$. If ϵ is sufficiently small then the edges w_1w_2, w_1w_3 and w_3w_4 have framework colour 1 and the edges w_1w_4, w_2w_3 and w_2w_4 have framework colour 2. This completes the first step. Every edge in G of the form uv_1 is replaced with uw_j for some j and if r is sufficiently small then the framework colours for these replacement edges remain the same. All other framework colours are unchanged and so G' is spanned by both monochrome subgraphs. $\qquad\square$

Theorem 4.9. *Let $\mathcal{P}$ be a polytope in $\mathbb{R}^2$ defined by a pair of linearly independent vectors $b_1, b_2 \in \mathbb{R}^2$ and let (G, p) be a well-positioned and regular bar-joint framework in $(\mathbb{R}^2, \|\cdot\|_{\mathcal{P}})$. The following statements are equivalent:*

    (1) *(G, p) is minimally infinitesimally rigid.*
    (2) *The monochrome subgraphs G_1 and G_2 induced by the framework colouring of (G, p) are edge-disjoint spanning trees.*
    (3) *G is $(2, 2)$-tight.*

Proof. (i) $\implies$ (ii). If (G, p) is minimally infinitesimally rigid then by Proposition 4.3 the monochrome subgraphs G_1 and G_2 each contain a spanning tree. If we remove a single edge from G then the corresponding framework is no longer infinitesimally rigid and so by Proposition 4.4 each monochrome subgraph G_k is itself a spanning tree.

(ii) $\implies$ (iii). The top count is clear since $|E| = |E(G_1)| + |E(G_2)| = 2n - 2$. If H is a subgraph of G then either $|V(H)| \leq 3$ or by adjoining edges to H we can build a minimally infinitesimally rigid framework (H', p). In either case we have $|E(H)| \leq 2|V(H)| - 2$.

(iii) $\implies$ (i). If G is $(2, 2)$-tight then there is a finite sequence of graph moves $K_1 \to G^{(1)} \to G^{(2)} \to \cdots \to G$ as described in Lemma 4.7. By repeated application of Lemma 4.8 we can choose well-positioned realisations $(G^{(j)}, p^{(j)})$ such that $G^{(j)}$ is spanned by both monochrome subgraphs $G_1^{(j)}$ and $G_2^{(j)}$. In particular, such a realisation exists for G and so by Proposition 4.4 G has a well-positioned and infinitesimally rigid realisation. All well-positioned regular realisations of G must also be infinitesimally rigid by Proposition 4.2 and so in particular (G, p) is infinitesimally rigid. Let $G = (V, E')$ be a graph obtained by removing a single edge from G. Then rank $R_q(G', p) \leq |E'| = 2n - 3$ and so by Proposition 4.2 (G', p) is not infinitesimally rigid. We conclude that (G, p) is minimally infinitesimally rigid. $\qquad\square$

5. Further directions

A general context for bar-joint frameworks is provided by a metric space $(\mathcal{M}, d)$ in which $\mathcal{M}$ is an embedded manifold in $\mathbb{R}^n$ and where the metric is obtained by restriction of a general norm $\|\cdot\|_C$. The case $n = 3$ and $\|\cdot\|_C = \|\cdot\|_2$ has been considered in [11][12] leading to Laman theorems for the circular cylinder, with two independent infinitesimal motions, and for surfaces of revolution, including the cone, the torus and elliptical cylinders, each of which has a single independent infinitesimal motion. Adopting a non-Euclidean norm such as $\|\cdot\|_q$ generally reduces the number of infinitesimal motions of the surface. In particular the $\|\cdot\|_q$-cylinder $\mathcal{M}_q = \{(x, y, z) : \|(x, y, 0)\|_q = 1\}$ has a 1-dimensional space of infinitesimal rigid motions. It is in fact possible to obtain a Laman theorem in this setting in terms of $(2, 1)$-tight graphs by arguing exactly as we have done here and employing the inductive characterisation in [12] of $(2, 1)$-tight graphs - the base graph is $K_5 \backslash e$ and with the four graph moves above there is an additional elementary graph move in which two graphs in the class are joined by a single edge. However, for the unit sphere $\mathcal{S}_q = \{(x, y, z) : \|(x, y, z)\|_q = 1\}$ there are no non-trivial rigid motions. Establishing a Laman theorem for frameworks which are vertex-supported by $\mathcal{S}_q$ along the lines above would require an inductive scheme for $(2, 0)$-tight graphs and this is not presently available.

References

[1] L. Asimow and B. Roth, The Rigidity of Graphs, Trans. Amer. Math. Soc., 245, (1978), 279-289.

[2] L. Asimow and B. Roth, The Rigidity of Graphs. II, J. Math. Anal. Appl., 68 (1979), no. 1, 171-190.

[3] Z. Fekete and L. Szego, A Note on $[k, l]$-sparse Graphs, in Graph Theory in Paris, Edited by A. Bondy et al, (2007), 169-177.

[4] H. Gluck, Almost All Simply Connected Closed Surfaces are Rigid, in Geometric Topology, Lecture Notes in Math., no. 438, Springer-Verlag, Berlin, 1975, pp. 225-239.

[5] J. Graver, B. Servatius, and H. Servatius, Combinatorial Rigidity, Graduate Studies in Mathematics, 2, American Mathematical Society, Providence, RI, 1993.

[6] B. Jackson, T. Jordán, The Dress Conjectures on Rank in the 3-Dimensional Rigidity Matroid, Adv. in Appl. Math., 35 (2005), no. 4, 355-367.

[7] G. Laman, On Graphs and the Rigidity of Plane Skeletal Structures, J. Engineering Mathematics, 4 (1970), 331-340.

[8] J. Lindenstrauss and L. Tzafriri, Classical Banach Spaces. I, Springer-Verlag, Berlin, 1977.

[9] C. St J.A. Nash-Williams, Decomposition of Finite Graphs into Forests, J. London Math. Soc. 39 (1964), 12.

[10] A. Nixon and J.C. Owen, An Inductive Construction of $(2, 1)$-tight Graphs, http://arxiv.org/abs/1103.2967v2, (2011).

[11] A. Nixon, J.C. Owen and S.C. Power, Rigidity of Frameworks Supported on Surfaces, SIAM J. Disc. Math., (2012).

[12] A. Nixon, J.C. Owen and S.C. Power, A Laman Theorem for Frameworks on Surfaces of Revolution. http://arxiv.org/abs/1210.7073, (2012).

[13] T-S. Tay, Rigidity of Multigraphs. I. Linking rigid bodies in n-space, J. of Combinatorial Th. Series B 36 (1984), 95-112.

[14] T-S. Tay, A New Proof of Laman's Theorem. Graphs and Combinatorics 9, (1993), 365-370.

[15] T-S. Tay, On the Generic Rigidity of Bar-Frameworks. Adv. in Appl. Math. 23 (1999), no. 1, 14-28.

[16] W. Whiteley, The Union of Matroids and the Rigidity of Frameworks, SIAM J. Disc. Math., Vol 1, no. 2, (1988), 237-255.

DEPT. MATH. STATS., LANCASTER UNIVERSITY, LANCASTER LA1 4YF, U.K.
E-mail address: d.kitson@lancaster.ac.uk, s.power@lancaster.ac.uk